Pismeni ispit iz Diferencijalne geometrije, 05.09.2014. (ispit pisati isključivo hemiskom olovkom plave ili crne tinte, obavezno navesti formulu koju koristite i značenje simbola iz napisane formule)

1. Data je kriva

$$L : x = a\cos^2 u,\ \ y = a\sin u \cos u,\ \ z = a\sin u$$

(a) Pokazati da kriva L leži u presjeku jedne lopte i cilindra čija je generatrisa paralelna osi $0z$ i odrediti jednačine tih površi.

(b) Odrediti jednačinu oskulatorne ravni krive L za $u = \dfrac{\pi}{2}$.

2. Data je kriva $\overrightarrow{r} = a\cos t\,\overrightarrow{i} + a\sin t\,\overrightarrow{j} + az(t)\,\overrightarrow{k}$. Odrediti $z(t)$ tako da kriva prolazi tačkama $(a, 0, a)$ i da njene tangente prodiru ravan $x0y$ u tačkama kružnice $x^2 + y^2 = R^2$.

3. Površ Γ definisana je vektorskom jednačinom

$$\vec{r} = (u\sin v,\ u\cos v,\ v).$$

Na površi je zadan krivoliniski trougao

$$0 \le u \le \operatorname{sh} v,\ \ 0 \le v \le v_0.$$

Izračunati uglove trougla.

4. Površ Γ definisana je vektorskom jednačinom

$$\vec{r} = (u\sin v,\ u\cos v,\ v).$$

(a) Naći prvu kvadratnu formu površi.

(b) Na površi je zadan krivoliniski trougao

$$0 \le u \le \operatorname{sh} v,\ \ 0 \le v \le v_0.$$

Izračunati površinu i dužine strana trougla.

Zadaci su skinuti sa stranice pf.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

(#) Data je kriva

$$L: \ x = a\cos^2 u, \quad y = a\sin u\cos u, \quad z = a\sin u$$

a) Pokazati da kriva L leži u presjeku jedne lopte i cilindra čija je generatrisa paralelna osi Oz i odrediti jednačine tih površi.

b) Odrediti jednačinu oskulatorne ravni krive L za $u = \frac{\pi}{2}$.

Rj. a) Primjetimo da je

$$x^2 + y^2 + z^2 = a^2\cos^4 u + a^2\sin^2 u\cos^2 u + a^2\sin^2 u =$$

$$= a^2\cos^2 u\underbrace{(\cos^2 u + \sin^2 u)}_{=1} + a^2\sin^2 u =$$

$$= a^2(\cos^2 u + \sin^2 u) = a^2$$

tj. $x^2 + y^2 + z^2 = a^2$ – jednačina sfere

i

$$x^2 + y^2 = a^2\cos^4 u + a^2\sin^2 u\cos^2 u = a^2\cos^2 u(\cos^2 u + \sin^2 u)$$

$$= a^2\cos^2 u = a \cdot \underbrace{a\cos^2 u}_{x} = ax$$

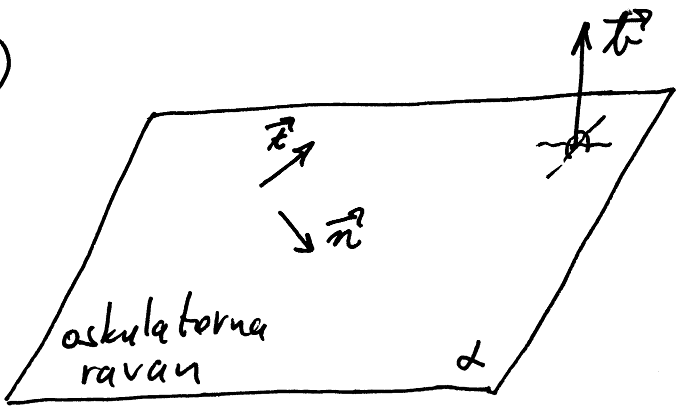
tj.

$x^2 + y^2 = ax$ – jednačina cilindrične površi čija je generatrisa paralelna osi Oz,

Vidimo da je kriva L određena presjekom ove dvije površi.

$$L: \begin{cases} x^2+y^2+z^2=a^2 \\ x^2+y^2=ax \end{cases}$$

b)



$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}}$$

$$\dot{\vec{r}} = \{-2a \sin u \cos u,\ a(\cos^2 u - \sin^2 u),\ a \cos u\}$$

$$\ddot{\vec{r}} = \{-2a \cos 2u,\ -2a \sin 2u,\ -a \sin u\}$$

Za $u=\frac{\pi}{2}$ je

$$\left.\begin{aligned} \dot{\vec{r}} &= (0,-a,0) \\ \ddot{\vec{r}} &= (2a,0,-a) \end{aligned}\right\} \Rightarrow \dot{\vec{r}} \times \ddot{\vec{r}} =$$

$$\vec{r} = (0,0,a)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -a & 0 \\ 2a & 0 & -a \end{vmatrix} =$$

$$= (a^2, 0, 2a^2)$$

Vidimo da je $M(0,0,a)$ proizvoljna tačka krive

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

$$a^2(x-0)+0\cdot(y-0)+2a^2(z-a)=0 \qquad /:a^2$$

$$x+2z-2a=0$$

je tražena jednačina oskulatorne ravni

Data je kriva $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + a\,z(t)\,\vec{k}$.
Odrediti $z(t)$ tako da kriva prolazi tačkama $(a,0,a)$ i da njene tangente prodiru ravan xOy u tačkama kružnice $x^2+y^2=R^2$.

Rj. $\vec{r} = (a\cos t, a\sin t, a\,z(t)) \Rightarrow$ za $t=0$ imamo $A(a,0,az(0))$.

Tačka $A(a,0,a)$ leži na krivoj, pa iz $az(0)=a$ slijedi $z(0)=1$.

Primjetimo da je jednačina tangente

$$\frac{x-a\cos t}{-\sin t} = \frac{y-a\sin t}{\cos t} = \frac{z-az(t)}{z'} \quad (=k)$$

Odredimo prodor tangente kroz ravan xOy

$$x - a\cos t = -k\sin t$$
$$y - a\sin t = k\cos t$$
$$z - az(t) = kz'$$

$$x = -k\sin t + a\cos t$$
$$y = k\cos t + a\sin t$$
$$z = kz' + az(t)$$

Za $k = -a\dfrac{z}{z'}$ ćemo dobiti da je $z=0$.
Prema tome prodor tangente kroz ravan xOy ima koordinate

$$Z = 0$$
$$X = a\frac{z}{z'}\sin t + a\cos t$$
$$Y = -a\frac{z}{z'}\cos t + a\sin t$$

Iz uslova $R^2 = X^2 + Y^2 = a^2\dfrac{z^2}{z'^2} + a^2$

dobija se $\dfrac{z^2}{z'^2} = \dfrac{R^2-a^2}{a^2}$ tj. $\dfrac{z'}{z} = \pm\sqrt{\dfrac{a^2}{R^2-a^2}}$, dakle

$$\ln|z| = \pm\sqrt{\frac{a^2}{R^2-a^2}}\,t + \ln|C|$$

Znači

$$z = Ce^{ut}, \quad \text{gdje je} \quad u = \pm\sqrt{\frac{a^2}{R^2-a^2}}.$$

No imali smo $z=a$ za $t=0$ (vidi točku $(a;0;a)$)

pa je $a=C$ tj.

$$z = a e^{\pm\sqrt{\frac{a^2}{R^2-a^2}}\,t} = a e^{ut},$$

traženo rješenje

(#) Površ Π definisana je vektorskom jednačinom

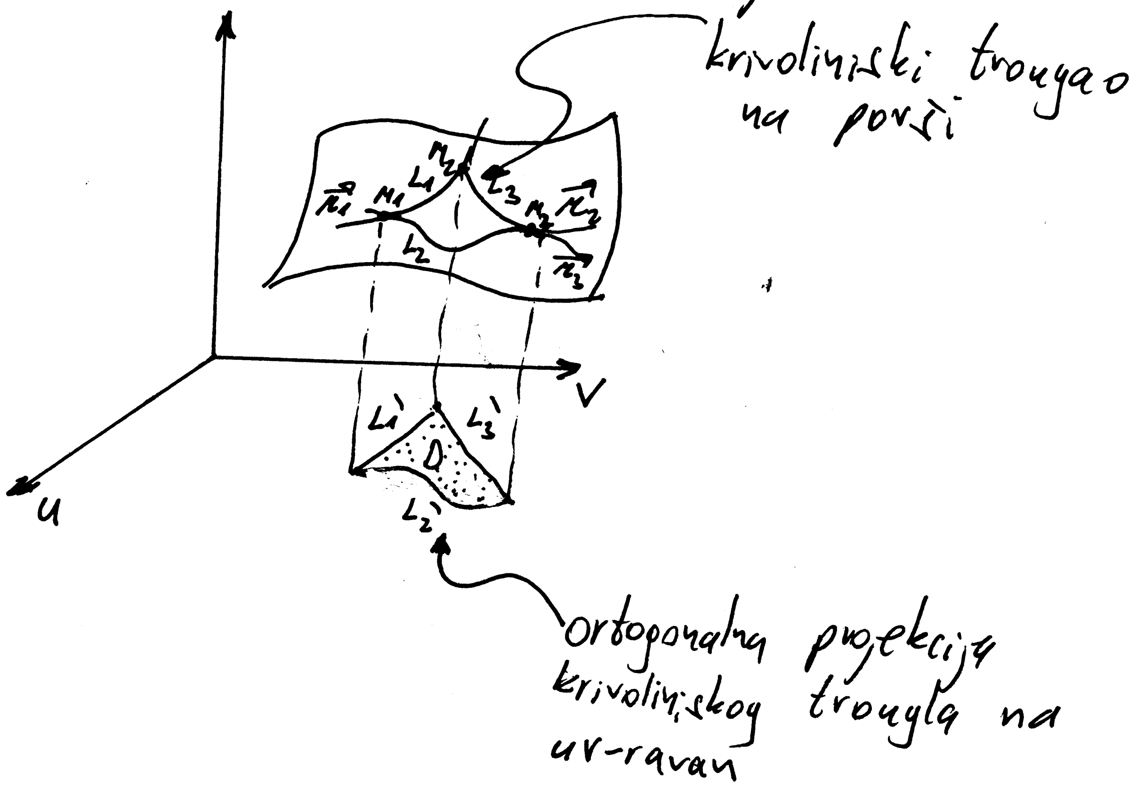
$$\vec{r} = (u\sin v,\ u\cos v,\ v).$$

Na površi je zadan krivolinijski trougao

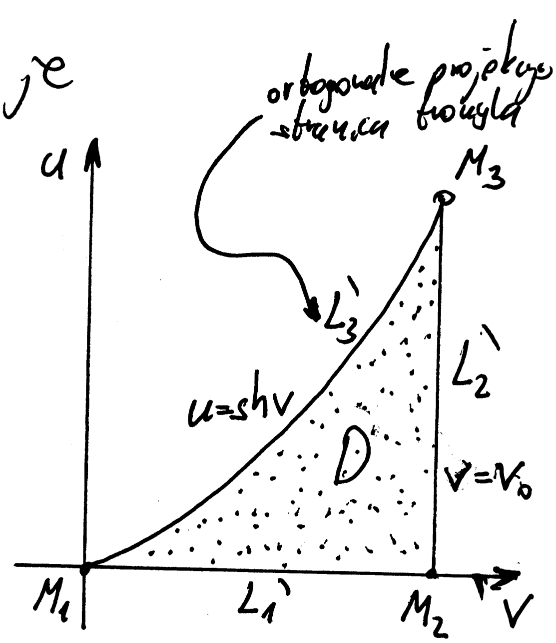
$$0 \leq u \leq \operatorname{sh} v, \quad 0 \leq v \leq v_0 .$$

Izračunati uglove trougla.

Rj. Posmatrajmo neku površ u opštem slučaju



U našem slučaju ortogonalna projekcija je



Odredimo jednačine stranica krivolinijskog trougla.

$$L_1' : \ u = 0 \ \xrightarrow{0 \leq v \leq v_0}$$

$$L_1 : \ \vec{r}_1 = (0, 0, v), \quad 0 \leq v \leq v_0 ,$$

$L_2': \ v = v_0, \ 0 \le u \le \operatorname{sh} v_0 \Rightarrow$

$$\Rightarrow \quad L_2: \vec{r}_2 = (u \sin v_0, u \cos v_0, v_0), \quad 0 \le u \le \operatorname{sh} v_0$$

$L_3': u = \operatorname{sh} v, \ 0 \le v \le v_0 \Rightarrow$

$$L_3: \vec{r}_3 = (\operatorname{sh} v \sin v, \operatorname{sh} v \cos v, v), \quad 0 \le v \le v_0$$

Odredimo presječne tačke krivih $\vec{r}_1, \vec{r}_2$ i $\vec{r}_3$

$$\left.\begin{array}{l} \vec{r}_1 = (0,0,v) \\ \vec{r}_2 = (u \sin v_0, u \cos v_0, v_0) \end{array}\right\} \Rightarrow \begin{array}{c} u \sin v_0 = 0 \\ u \cos v_0 = 0 \\ v_0 = v \end{array} \Rightarrow M_1(\overset{v}{v_0}, \overset{u}{0})$$

$$\left.\begin{array}{l} \vec{r}_1 = (0,0,v) \\ \vec{r}_3 = (\operatorname{sh} v \sin v, \operatorname{sh} v \cos v, v) \end{array}\right\} \Rightarrow \begin{array}{c} \operatorname{sh} v \sin v = 0 \\ \operatorname{sh} v \cos v = 0 \end{array} \Rightarrow M_2(\overset{v}{0}, \overset{u}{0})$$

slično za $\vec{r}_2$ i $\vec{r}_3$ $\quad M_3(\overset{v}{v_0}, \overset{u}{\operatorname{sh} v_0})$

<u>Ugao između dvije krive linije se definiše kao ugao između tangenti u presječnoj tački tih krivih.</u>

$\dot{\vec{r}}_1 = (0,0,1)$

$\dot{\vec{r}}_2 = (\sin v_0, \cos v_0, 0)$

$\dot{\vec{r}}_3 = (\operatorname{ch} v \sin v + \operatorname{sh} v \cos v, \operatorname{ch} v \cos v - \operatorname{sh} v \sin v, 1)$

Znamo da je

$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos \sphericalangle(\vec{x}, \vec{y})$$

$$\cos \sphericalangle(\vec{x}, \vec{y}) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$$

Time je

$$\cos \varphi_1 = \left. \frac{\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2}{|\dot{\vec{r}}_1||\dot{\vec{r}}_2|} \right|_{M_1} = \overset{\text{ZA VJEŽBU}}{\ldots} = 0 \Rightarrow \varphi_1 = \frac{\pi}{2}$$

$$\cos \varphi_2 = \cos \varphi_3 = \ldots = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \varphi_2 = \varphi_3 = \frac{\pi}{4}$$

traženi uglovi

\# Površ Γ definisana je vektorskom jednačinom

$$\vec{r} = (u \sin v,\ u \cos v,\ v)$$

a) Naći prvu kvadratnu formu površi.

b) Na površi je zadan krivolinijski trougao

$$0 \le u \le \operatorname{sh} v, \quad 0 \le v \le v_0$$

Izračunati površinu, dužine strana i uglove trougla.

Rj:

a) Prva kvadratna forma površi je

$$F_1 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

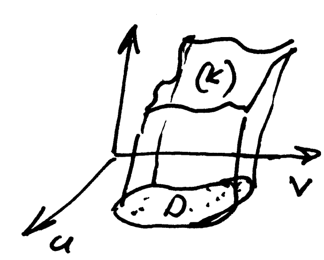
$$E = (\vec{r}\,'_u)^2, \quad F = (\vec{r}\,'_u \cdot \vec{r}\,'_v), \quad G = (\vec{r}\,'_v)^2$$

$$\left.\begin{array}{l} \vec{r}\,'_u = (\sin v,\ \cos v,\ 0) \\ \vec{r}\,'_v = (u\cos v,\ -u\sin v,\ 1) \end{array}\right\} \Rightarrow \begin{array}{l} E = 1 \\ F = 0 \\ G = 1+u^2 \end{array}$$

Prva kvadratna forma je

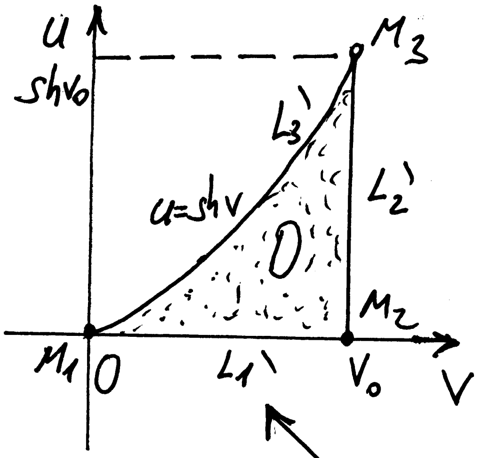
$$F_1 = ds^2 = du^2 + (1+u^2)\,dv^2$$

b) Površinu dijela površi računamo po formuli:



$$P = \iint\limits_{(K)} dS = \iint\limits_{D} \sqrt{EG-F^2}\,du\,dv$$

Oblast D nije teško skicirati



ortogonalne projekcije stranica trougla

$$D: \begin{cases} 0 \le u \le \operatorname{sh} v \\ 0 \le v \le v_0 \end{cases}$$

U našem slučaju je

$$P = \iint_D \sqrt{1+u^2}\,du\,dv = \int_0^{v_0} dv \int_0^{\operatorname{sh} v} \sqrt{1+u^2}\,du \stackrel{(*)}{=}$$

$$\int \sqrt{1+u^2}\,du = \begin{matrix}\text{ZA}\\ \text{VJEŽBU}\\ \dots\end{matrix} = \frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)$$

$$\int_0^{\operatorname{sh} v} \sqrt{1+u^2}\,du = \frac{u}{2}\sqrt{1+u^2}\Big|_0^{\operatorname{sh} v} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\Big|_0^{\operatorname{sh} v} =$$

$$= \frac{1}{2}\operatorname{sh} v\sqrt{1+\operatorname{sh}^2 v} + \frac{1}{2}\ln\left(\operatorname{sh} v + \sqrt{1+\operatorname{sh}^2 v}\right) =$$

(Znamo da je $1+\operatorname{sh}^2 v = \operatorname{ch}^2 v$)

$\operatorname{ch} v \ge 0 \quad \forall v \in \mathbb{R}$

$$= \frac{1}{2}\operatorname{sh} v|\operatorname{ch} v| + \frac{1}{2}\ln(\operatorname{sh} v + \operatorname{ch} v)$$

$$= \frac{1}{2}\operatorname{sh} v \cdot \operatorname{ch} v + \frac{1}{2}\ln(\operatorname{sh} v + \operatorname{ch} v)$$

$$\stackrel{(*)}{=} \int_0^{v_0}\left(\frac{1}{2}\operatorname{sh} v \operatorname{ch} v + \frac{1}{2}\ln(\operatorname{sh} v + \operatorname{ch} v)\right)dv = \underbrace{\frac{1}{2}\int_0^{v_0}\operatorname{sh} v \operatorname{ch} v\,dv}_{I_1} + \underbrace{\frac{1}{2}\int_0^{v_0}\ln(\operatorname{sh} v + \operatorname{ch} v)\,dv}_{I_2}$$

$$I_1 = \frac{1}{2}\int_0^{v_0}\operatorname{sh} v \operatorname{ch} v\,dv = \frac{1}{2}\int_0^{v_0}\operatorname{sh} v\,d(\operatorname{sh} v) = \frac{1}{2}\cdot\frac{1}{2}\operatorname{sh}^2 v\Big|_0^{v_0} = \frac{1}{4}\operatorname{sh}^2 v$$

$$\operatorname{sh} x + \operatorname{ch} x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$$

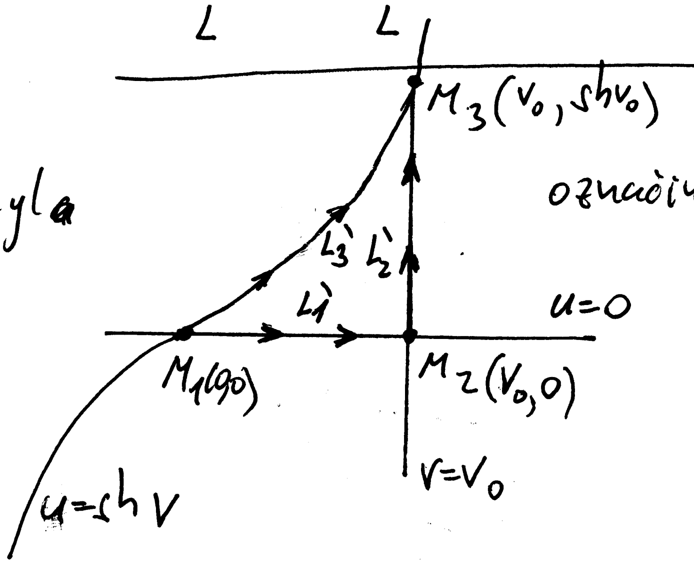
$$I_2 = \frac{1}{2}\int_0^{v_0} \ln(e^v)\,dv = \frac{1}{2}\int_0^{v_0} v\,dv = \frac{1}{4}v^2\Big|_0^{v_0} = \frac{1}{4}v_0^2$$

Prema tome $\quad P = \frac{1}{4}\left(v_0^2 + sh^2 v_0\right)$ tražena površina

Dužina luka L se računa po obrascu

$$s = \int_L ds = \int_L \sqrt{du^2 + (1+u^2)\,dv^2}$$

Dužine stranica trougla označimo sa s_1, s_2 i s_3



$$s_1 = \int_{L_1} ds = \int_{L_1} \sqrt{du^2 + (1+u^2)dv^2} = \int_0^{v_0} dv = v_0$$

$$s_2 = \int_{L_2} \sqrt{du^2 + (1+u^2)\,dv^2} = \left| \begin{array}{l} L_2 : v = v_0 \\ dv = 0 \end{array} \right| = \int_0^{shv_0} du = sh\,v_0$$

$$s_3 = \int_{L_3} \sqrt{du^2 + (1+u^2)dv^2} = \left| \begin{array}{l} u = sh\,v \\ du = ch\,v\,dv \\ du^2 = ch^2 v\,dv^2 \end{array} \right| = \int_0^{v_0} \sqrt{ch^2 v + 1 + sh^2 v}\,dv =$$

$$= \left| \begin{matrix} \text{znamo da} \\ 1+\operatorname{sh}^2 v = \operatorname{ch}^2 v \end{matrix} \right| = \sqrt{2}\int_0^{v_0} \operatorname{ch} v \, dv = \sqrt{2}\, \operatorname{sh} v_0$$

Dužine strana krivoliniskog trougla na datoj površini su

$$s_1 = v_0, \qquad s_2 = \operatorname{sh} v_0, \qquad s_3 = \sqrt{2}\, \operatorname{sh} v_0$$

Jednačine stranica krivoliniskog trougla su

$L_1: \vec{r}_1 = (0, 0, v), \quad 0 \le v \le v_0$

$L_2: \vec{r}_2 = (u \sin v_0, u \cos v_0, v_0), \quad 0 \le u \le \operatorname{sh} v_0$

$L_3: \vec{r}_3 = (\operatorname{sh} v \sin v, \operatorname{sh} v \cos v, v), \quad 0 \le v \le v_0$

Presječna tačka stranica L_1 i L_2 je $M_1(v_0, 0)$, stranica L_1 i L_3 je $M_2(0,0)$ i L_2 i L_3 je $M_3(v_0, \operatorname{sh} v_0)$

Stoga je $\quad \cos\varphi_1 = \left.\dfrac{\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2}{|\dot{\vec{r}}_1||\dot{\vec{r}}_2|}\right|_{M_1} \overset{\text{za vježbu}}{=} 0$

$$\cos\varphi_2 = \left.\frac{\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_3}{|\dot{\vec{r}}_1||\dot{\vec{r}}_3|}\right|_{M_2} = \ldots^{\text{za vježbu}} = \frac{1}{\sqrt{2}} = \left.\frac{\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_3}{|\dot{\vec{r}}_2| \cdot |\dot{\vec{r}}_3|}\right|_{M_3} = \cos\varphi_3$$

i $\varphi_1 = \dfrac{\pi}{2}, \quad \varphi_2 = \varphi_3 = \dfrac{\pi}{4}$ traženi uglovi